A HORIZONTAL GYROCOMPASS ON VIBRATING ELASTIC FOUNDATION

(GIROGORIZONTKOMPAS NA VIBRIRUIUSHCHEM OSNOVANII)

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The influence of vibrations on the readings of a horizontal gyrocompass with two rotors is investigated. The inertia forces of the elements of the compass and the elastic properties of the gyroscopes' membranes are being taken into account. The formulas obtained permit the calculation of the most dangerous (resonance) vibration frequencies and the amount of turning of the compass through an azimuth angle as a function of the frequency.

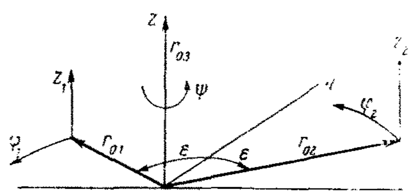

Fig. 1.

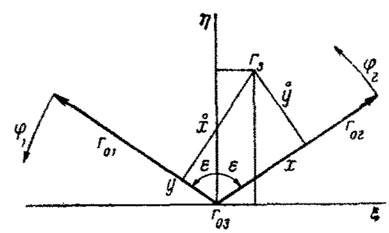

Fig. 2.

The influence of vibrations is expressed through the external periodic moments M_ξ^* and M_η^* in the horizontal plane, about the eastern and the northern axes of the instrument, respectively. These and other moments can be transmitted to the gyroscopes' rotors only through the deformations of the membranes. The axis of symmetry of a rotor r and the axis of symmetry of its casing R, usually coinciding, diverge when the membrane is deformed. The moment of the elastic forces acting on a rotor is expressed by the formula

$$\mathbf{M} = \mu \mathbf{r} \times \mathbf{R}$$

Here **r** and **R** are unit vectors, μ is the transverse rigidity of a membrane.

The mean orientation of the vibrating elements of the compass, that is, of its two rotors and of the shell, is determined by three unit vectors:

$\mathbf{r}_{01}$, the unit vector of the axis of symmetry of the first rotor;

$\mathbf{r}_{02}$, the unit vector of the axis of symmetry of the second rotor;

$\mathbf{r}_{03}$, the unit vector of the axis of the shell, which is parallel to the rotation axes of the casings (the so-called axes of precession).

An orientation at an arbitrary deflection is determined by unit vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, respectively, and by an angle ψ which is the rotation angle of the shell about $\mathbf{r}_3$. At an arbitrary deflection of the shell and of the rotors, the axes of the casings remain perpendicular to the axis of precession $\mathbf{r}_3$. The axes of the casings will turn about $\mathbf{r}_3$ relative to the shell through angles χ which equal each other because of the sectorial constraint. The orientation of the axes of the casings after this turning will be determined by the unit vectors $\mathbf{R}_1$ and $\mathbf{R}_2$.

The vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ are uniquely determined through the vectors $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$ and the angle ψ.

In deriving the equations of motion, we shall use the coordinate system shown in Figs. 1 and 2, where ($\mathbf{r}_{01}$, $\mathbf{r}_{02}$) is the equatorial plane. Let ϕ_1 and ϕ_2 be angular deviations of $\mathbf{r}_1$ and $\mathbf{r}_2$ from their mean position in the equatorial plane; let z_1 and z_2 be the deviations of $\mathbf{r}_1$ and $\mathbf{r}_2$ above the equatorial plane; let x and y be the projections of the deviated vector $\mathbf{r}_3$ on $\mathbf{r}_{02}$ and $\mathbf{r}_{01}$. Let ψ be the rotation angle of the shell about $\mathbf{r}_3$. The orientation of the vector $\mathbf{r}_3$ determined through x and y can also be determined through the coordinates ξ, η, or x°, y°, which is explained in Fig. 2.

The equations of motion representing the law of the rate of change of the angular momentum for each of the three bodies, that is, for the two rotors and for the shell, have the form

$$\begin{aligned} &H\dot{z}_1 + A\ddot{\varphi}_1 = M_{z1}, && I\ddot{\xi} = M_\eta + M_\eta{}^* \\ &H\dot{z}_2 + A\ddot{\varphi}_2 = M_{z_2}, && I\ddot{\eta} = -M_\xi - M_\xi{}^* \\ &H\dot{\varphi}_1 - A\ddot{z}_1 = M_{\varphi_1}, && I\ddot{\psi} = M_\psi \\ &H\dot{\varphi}_2 - A\ddot{z}_2 = M_{\varphi_2} \end{aligned} \tag{2}$$

Here H is the angular momentum of a rotor about its rotation axis, A is the moment of inertia of a rotor about the axis of precession, I is the moment of inertia of the shell. The right-hand terms in the equations represent the projections on the respective axes of the elastic moments generated by the membranes. The moments M_ξ^* and M_η^*, as mentioned previously, are the only exterior moments connected with vibrations of the foundation.

When deriving (2) we neglected the rigidity of the spring constraint and of the pendular properties of the instruments, since they correspond to rigidities considerably smaller than μ. The ellipsoid of inertia of the shell is assumed to be a sphere and the gyroscopic effects of the shell are neglected. In the formulas for the elastic moments, which will follow, the inertia of the casings is also neglected.

The elastic moments of the membranes $\mathbf{M}_1$ and $\mathbf{M}_2$ acting on the first and on the second rotor, respectively, are, on the strength of (1), given by

$$\mathbf{M}_1 = \mu\,(\mathbf{r}_1 \times \mathbf{R}_1), \qquad \mathbf{M}_2 = \mu\,(\mathbf{r}_2 \times \mathbf{R}_2) \tag{3}$$

Since

$$\mathbf{M}_1 \cdot (\mathbf{r}_2 \times \mathbf{r}_1) = \mu\,(\mathbf{r}_1 \cdot \mathbf{R}_1)(\mathbf{r}_1 \cdot \mathbf{r}_3), \qquad \mathbf{M}_2 \cdot (\mathbf{r}_3 \times \mathbf{r}_2) = \mu\,(\mathbf{r}_2 \cdot \mathbf{R}_2)(\mathbf{r}_2 \cdot \mathbf{r}_3) \tag{4}$$

therefore, with sufficient accuracy

$$\begin{aligned} \mathbf{M}_1 &= \mu\,(\mathbf{r}_1 \cdot \mathbf{R}_1)(\mathbf{r}_1 \cdot \mathbf{r}_3)(\mathbf{r}_3 \times \mathbf{r}_1) + \mu\mathbf{r}_3 \cdot (\mathbf{r}_1 \times \mathbf{R}_1)\,\mathbf{r}_3 \\ \mathbf{M}_2 &= \mu\,(\mathbf{r}_2 \cdot \mathbf{R}_2)(\mathbf{r}_2 \cdot \mathbf{r}_3)(\mathbf{r}_3 \times \mathbf{r}_2) + \mu\mathbf{r}_3 \cdot (\mathbf{r}_2 \times \mathbf{R}_2)\,\mathbf{r}_3 \end{aligned} \tag{5}$$

In our coordinate system

$$\begin{aligned} \mathbf{r}_1 \cdot \mathbf{r}_3 &= z_1 + y, \qquad & \mathbf{r}_3 \cdot (\mathbf{r}_1 \times \mathbf{R}_1) &= \psi - \chi - \varphi_1 \\ \mathbf{r}_2 \cdot \mathbf{r}_3 &= z_2 + x, \qquad & \mathbf{r}_3 \cdot (\mathbf{r}_2 \times \mathbf{R}_2) &= \psi + \chi - \varphi_2 \end{aligned} \tag{6}$$

The condition of equilibrium for the casings, neglecting inertia, requires $\mathbf{M}_1 \cdot \mathbf{r}_3 = \mathbf{M}_2 \cdot \mathbf{r}_3$; hence

$$\chi = \frac{1}{2}(\varphi_2 - \varphi_1) \tag{7}$$

Since the deformation is small we can set

$$\mathbf{R}_1 \cdot \mathbf{r}_1 = 1, \qquad \mathbf{R}_2 \cdot \mathbf{r}_2 = 1 \tag{8}$$

On the strength of (5) and taking into account (6), (7), and (8), we have

$$\begin{aligned} \mathbf{M}_1 &= \mu\,(z_1 + y)(\mathbf{r}_1 \times \mathbf{r}_3) + \mu\left(\psi - \frac{\varphi_1 + \varphi_2}{2}\right)\mathbf{r}_3 \\ \mathbf{M}_2 &= \mu\,(z_2 + x)(\mathbf{r}_2 \times \mathbf{r}_3) + \mu\left(\psi - \frac{\varphi_1 + \varphi_2}{2}\right)\mathbf{r}_3 \end{aligned} \tag{9}$$

Writing down the scalar components along the respective axes of the vector equations (9) and neglecting terms of the third order of smallness, we obtain

$$
\begin{aligned}
M_{\varphi_1} &= \mu\,(z_1 + y) + x^\circ\mu\left(\frac{\varphi_1+\varphi_2}{2} - \psi\right)\\
M_{\varphi_2} &= \mu\,(z_2 + \varkappa) - y^\circ\mu\left(\frac{\varphi_1+\varphi_2}{2} - \psi\right)\\
M_{z_1} &= \mu\left(\psi - \frac{\varphi_1+\varphi_2}{2}\right) + x^\circ\mu\,(z_1 + y)\\
M_{z_2} &= \mu\left(\psi - \frac{\varphi_1+\varphi_2}{2}\right) - y^\circ\mu\,(z_2 + x)
\end{aligned}
\tag{10}
$$

The elastic moment acting on the shell will be denoted by M_0. It is clear that

$$\mathbf{M}_0 = -\mathbf{M}_1 - \mathbf{M}_2.$$

Writing down the scalar components along the respective axes of the above vector equation and neglecting terms of the second order of smallness, we obtain

$$
\begin{aligned}
M_\xi &= \mu\,(z_1 + y + z_2 + x)\cos\varepsilon\\
M_\eta &= \mu\,(z_1 + y - z_2 - x)\sin\varepsilon\\
M_\psi &= 2\mu\left(\frac{\varphi_1+\varphi_2}{2} - \psi\right)
\end{aligned}
\tag{11}
$$

From Fig. 2 we have

$$
\begin{aligned}
x &= \eta\cos\varepsilon + \xi\sin\varepsilon, \qquad x^\circ = \eta\sin\varepsilon + \xi\cos\varepsilon\\
y &= \eta\cos\varepsilon - \xi\sin\varepsilon, \qquad y^\circ = \eta\sin\varepsilon - \xi\cos\varepsilon
\end{aligned}
\tag{12}
$$

It follows that

$$
\begin{aligned}
\xi &= \frac{x-y}{2\sin\varepsilon}, \qquad x^\circ = \frac{x+y}{2}\tan\varepsilon + \frac{x-y}{2}\cot\varepsilon\\
\eta &= \frac{x-y}{2\cos\varepsilon}, \qquad y^\circ = \frac{x+y}{2}\tan\varepsilon - \frac{x-y}{2}\cot\varepsilon
\end{aligned}
\tag{13}
$$

The relations (13) permit all the quantities appearing in Formulas (2), (10) and (11) to be expressed in terms of the coordinates ϕ_1, ϕ_2, z_1, z_2, x, y, ψ. The linear part of the system (2) takes the form

$$
\begin{aligned}
&H\dot z_1 + A\ddot\varphi_1 = \mu\left(\psi - \frac{\varphi_1+\varphi_2}{2}\right), \qquad H\dot\varphi_1 - A\ddot z_1 = \mu\,(z_1 + y)\\
&H\dot z_2 + A\ddot\varphi_2 = \mu\left(\psi - \frac{\varphi_1+\varphi_2}{2}\right), \qquad H\dot\varphi_2 - A\ddot z_2\mu = (z_2 + x)\\
&I\frac{\ddot x - \ddot y}{2\sin\varepsilon} = \mu\sin\varepsilon\,(z_1 + y - z_2 - x) + M_\eta^*, \qquad I\ddot\psi = 2\mu\left(\frac{\varphi_1+\varphi_2}{2} - \psi\right)\\
&I\frac{\ddot x + \ddot y}{2\cos\varepsilon} = -\mu\cos\varepsilon\,(z_1 + y + z_2 + x) - M_\xi^*
\end{aligned}
\tag{14}
$$

The nonlinear parts of the moments (10) (they will be denoted by δM_{ϕ_1}, δM_{ϕ_2}, δM_{z_1}, δM_{z_2}) may have constant components ΔM_{ϕ_1}, ΔM_{ϕ_2}, ΔM_{z_1}, ΔM_{z_2}, which are equivalent to the external moments ΔM_ξ, ΔM_η, ΔM_z and to the moment ΔN resisting the spring moment $N(\epsilon)$. Thus

$$\Delta M_\xi = -\cos\varepsilon\,(\Delta M_{\varphi_1} + \Delta M_{\varphi_2}), \qquad \Delta M_z = \Delta M_{z_1} + \Delta M_{z_2}$$
$$\Delta M_\eta = -\sin\varepsilon\,(\Delta M_{\varphi_1} - \Delta M_{\varphi_2}), \qquad \Delta N = \Delta M_{z_1} - \Delta M_{z_2} \tag{15}$$

The most important resisting moment is ΔM_z. Taking into account (13) we obtain for M_z the following expression:

$$\delta M_z = \frac{\mu}{\sin 2\varepsilon}\{\sin^2\varepsilon\,(x+y)(z_1+y-z_2-x) + \cos^2\varepsilon\,(x-y)(z_1+y+z_2+x)\} \tag{16}$$

In order to investigate the system (14) we introduce the variables

$$\alpha = \varphi_1 + \varphi_2, \quad \beta = z_1 + z_2, \quad \gamma = z_1 - z_2, \quad \delta = \varphi_1 - \varphi_2$$
$$\alpha_1 = \psi, \quad \beta_1 = x + y, \quad \gamma_1 = x - y \tag{17}$$

By suitable additions and subtractions the equations in (14) can be transformed into the separable system

$$\begin{gathered}
\frac{I}{\mu}\ddot{\gamma}_1 + 2\sin^2\varepsilon\gamma_1 - 2\sin^2\varepsilon\gamma = \frac{2}{\mu}\sin\varepsilon\, M_\eta^*\\
\frac{H}{\mu}\dot{\delta} - \frac{A}{\mu}\ddot{\gamma} = \gamma - \gamma_1, \qquad H\dot{\gamma} + A\ddot{\delta} = 0\\
\frac{I}{\mu}\ddot{\beta}_1 + 2\cos^2\varepsilon\beta_1 + 2\cos^2\varepsilon\beta = -\frac{2}{\mu}\cos\varepsilon\, M_\xi^*\\
\frac{H}{\mu}\dot{\alpha} - \frac{A}{\mu}\ddot{\beta} = \beta + \beta_1, \qquad \frac{H}{\mu}\dot{\beta} + \frac{A}{\mu}\ddot{\alpha} = 2\alpha_1 - \alpha\\
\frac{I}{\mu}\ddot{\alpha}_1 + 2\alpha_1 - \alpha = 0
\end{gathered} \tag{18}$$

Eliminating δ and using matrix notation we have

$$\begin{Vmatrix} 2\sin^2\varepsilon + \frac{I}{\mu}D^2 & -2\sin^2\varepsilon \\ -1 & 1 + \frac{H^2}{\mu A} + \frac{A}{\mu}D^2 \end{Vmatrix} \begin{Vmatrix} \gamma_1 \\ \gamma \end{Vmatrix} = \begin{Vmatrix} \frac{2\sin\delta}{\mu} M_\eta^* \\ 0 \end{Vmatrix} \tag{19}$$

$$\left\|\begin{matrix} 2\cos^2\varepsilon + \frac{I}{\mu}D^2 & 2\cos^2\varepsilon & 0 & 0 \\ 1 & 1+\frac{A}{\mu}D^2 & -\frac{H}{\mu}D & 0 \\ 0 & \frac{H}{\mu}D & 1+\frac{A}{\mu}D^2 & -2 \\ 0 & 0 & -1 & 2+\frac{I}{\mu}D^2 \end{matrix}\right\| \left\|\begin{matrix}\beta_1\\ \beta\\ \alpha\\ \alpha_1\end{matrix}\right\| = \left\|\begin{matrix} -\frac{2\cos\varepsilon}{\mu}M_\xi^* \\ 0 \\ 0 \\ 0\end{matrix}\right\| \tag{20}$$

When the moments M_ξ^* and M_η^* are sinusoidal with the frequency ω, then the solution of (19) and (20) yields

$$\gamma_1 = \frac{\Delta_{\gamma_1}}{\Delta_1} 2\sin\varepsilon\, M_\eta^*, \qquad \beta_1 = -\frac{\Delta_{\beta_1}}{\Delta_2} 2\cos\varepsilon\, M_\xi^*$$
$$\gamma = \frac{\Delta_{\gamma}}{\Delta_1} 2\sin\varepsilon\, M_\eta^*, \qquad \beta = -\frac{\Delta_{\beta_2}}{\Delta_2} 2\cos\varepsilon\, M_\xi^* \tag{21}$$

Here Δ are the corresponding minors and determinants when $D = i\omega$. Substituting (17) and (21) into (16), we obtain the expression

$$\delta M_z = -\frac{2M_\xi^* M_\eta^*}{\mu\Delta_1\Delta_2}\{\cos 2\varepsilon\,\Delta_{\beta_1}\Delta_{\gamma_1} + \cos^2\varepsilon\,\Delta_\beta\Delta_{\gamma_1} + \sin^2\varepsilon\,\Delta_\beta\Delta_\gamma\} \tag{22}$$

If

$$M_\xi^* = M\sin\theta\,\sin\omega t \qquad M_\eta^* = M\cos\theta\,\sin(\omega t + \vartheta) \tag{23}$$

then ΔM_z reaches a maximum when $\theta = 1/4\,\pi$, $\vartheta = 0$, and

$$M_z^* = \max\Delta M_z = -\frac{M^2}{2\mu\Delta_1\Delta_2}\{\cos 2\varepsilon\,\Delta_{\beta_1}\Delta_{\gamma_1} + \cos^2\varepsilon\,\Delta_\beta\Delta_{\gamma_1} + \sin^2\varepsilon\,\Delta_{\beta_1}\Delta_\gamma\} \tag{24}$$

Calculations give

$$\Delta_1 = 2\sin^2\varepsilon - \lambda\{1 + s\varkappa(1 + 2\varkappa\sin^2\varepsilon)\} + s\varkappa^2\lambda^2$$
$$\Delta_{\gamma_1} = (1 + s\varkappa) - s\varkappa^2\lambda, \qquad \Delta_\gamma = s\varkappa\, 2\sin^2\varepsilon \tag{25}$$

$$\Delta_2 = 4\cos^2\varepsilon - \lambda\{2(1+\cos^2\varepsilon) + s[1 - 2\varkappa(1+\cos^2\varepsilon) + 4\varkappa^2\cos^2\varepsilon]\} + \lambda^2\{1 + 2s\varkappa[1 + \varkappa(1+\cos^2\varepsilon)]\} - s\varkappa^2\lambda^3 \tag{26}$$
$$\Delta_{\beta_1} = 2 + s(1+2\varkappa) - \lambda\{1 + 2s\varkappa(1+\varkappa)\} + s\varkappa^2\lambda^2$$
$$\Delta_\beta = s(1+2\varkappa) - s\varkappa\lambda$$

Here

$$s = \frac{I\mu}{H^2}, \qquad \varkappa = \frac{A}{I}, \qquad \lambda = \frac{I\omega^2}{\mu} \tag{27}$$

The roots of the equation

$$\Delta_1(\lambda)\ \Delta_2(\lambda) = 0 \tag{28}$$

give the resonance frequencies. By Formulas (24) to (28) we can calculate both the resonance frequencies and the quantity M_z^* at any vibration frequency ω.

Equation (28) has five roots. The elasticity of the membranes determines, in principle, the three smaller roots; the two larger roots are connected with higher-order terms of the polynomials $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$. The latter two roots can be very roughly approximated by the formula

$$\lambda_4 \approx \lambda_5 \approx 1/s\varkappa$$

and they correspond to the frequency

$$\omega_{4,5} = \frac{H}{A}$$

which is the nutation frequency. Nutation frequencies are, as a rule, quite high, therefore, when the low frequency spectrum of vibrations is being investigated, the higher-order terms of the polynomials $\Delta_1(\lambda)$ and $\Delta_2(\lambda)$ can obviously be neglected. It should be mentioned, though, that the presence of the high-order terms in Formulas (25) and (26) does not cause any special difficulties.

When the elasticity of the sectorial constraints of the casings is taken into account, then the relations (9) will change a little and assume the form

$$\begin{aligned} \mathbf{M}_1 &= \mu\,(z_1 + y)\,\mathbf{r}_1 \times \mathbf{r}_3 + \mu_0\left(\psi - \frac{\varphi_1 + \varphi_2}{2}\right)\mathbf{r}_3 \\ \mathbf{M}_2 &= \mu\,(z_2 + x)\,\mathbf{r}_2 \times \mathbf{r}_3 + \mu_0\left(\psi - \frac{\varphi_1 + \varphi_2}{2}\right)\mathbf{r}_3 \end{aligned} \tag{29}$$

Denoting the rigidity of the elastic constraint of the bisector of the axes of the casings and of the northern axis of the shell by μ_1, we have

$$\mu_0 = \frac{1}{\dfrac{1}{\mu} + \dfrac{2}{\mu_1}}, \qquad \text{or} \qquad \frac{\mu}{\mu_0} = K = 1 + \frac{2\mu}{\mu_1} \tag{30}$$

Formula (24) for the resistance M_z^* and the formulas in (25) remain valid, but the relations in (26) must be replaced by

$$\begin{aligned} \Delta_2 &= 4\cos^2\varepsilon - \lambda\,\{2\,(1 + K\cos^2\varepsilon) + s\,[1 + 2\varkappa\,(1 + \cos^2\varepsilon) + 4\varkappa^2\cos^2\varepsilon]\} + \\ &\quad + \lambda^2\,\{K + s\varkappa\,[(1 + 2\varkappa) + K\,(1 + 2\varkappa\cos^2\varepsilon)]\} - Ks\varkappa^2\lambda^3 \\ \Delta_{\beta_1} &= 2 + s\,(1 + 2\varkappa) - \lambda\,\{K + s\varkappa\,(1 + K + 2\varkappa)\} + Ks\varkappa^2\lambda^2 \\ \Delta_\beta &= (1 + 2\varkappa) - K\varkappa\lambda \end{aligned} \tag{31}$$

When $K = 1$, which corresponds to $\mu_1 = \infty$, then the relations (31) reduce, as they should, to (26).

To conclude, we present a numerical example. Let the parameters of a gyrocompass have the following values:

$H = 10^5$ g cm sec; $I = 500$ g cm sec^2; $A = 30$ g cm sec^2;

$\mu = 6 \times 10^6$ g cm; $Pl = 4500$ g cm

The angle between the axes of the rotors $\epsilon = 60°$; the amplitudinal acceleration of the vibrations $w = 0.1$ g; the latitude $\phi = 60°$.

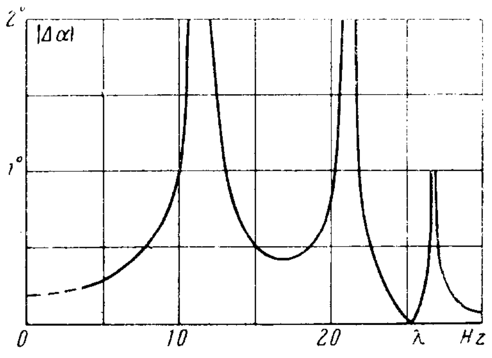

Fig. 3.

These parameters correspond to

$$M = Plw/g = 450 \text{ g cm}$$

From (27) we have

$$s = 0.3 \qquad \varkappa = 0.06, \qquad \lambda = \frac{\omega^2}{1.2 \cdot 10^4}$$

From Formulas (25) and (26) we find

$$\Delta_1 = 1.5 - 1.02\ \lambda + 0.0011\ \lambda^2$$
$$\Delta_{\gamma_1} = 1.018 - 0.0011\ \lambda$$
$$\Delta_\gamma = 0.027$$
$$\Delta_2 = 1 - 2.85\ \lambda + 1.04\ \lambda^2 - 0.0011\ \lambda$$
$$\Delta_{\beta_1} = 2.336 - 1.04\lambda + 0.0011\ \lambda^2$$
$$\Delta_\beta = 0.336 - 0.018\ \lambda$$

When the values of λ are not too large, then we derive from (24) quite an accurate formula for M_z^*

$$M_z^* = 0.087 \frac{M^2}{\mu} \frac{2.1 - \lambda}{(1.47 - \lambda)(0.35 - \lambda + 0.37\lambda^2)}$$

The moment M_z^* causes the gyrocompass to turn through the azimuth angle $\Delta\alpha$

$$\Delta\alpha = \frac{M_z^*}{2H \cos \varepsilon U \cos \varphi} = \frac{M_z^*}{3.6 \text{ g cm}}$$

The above formulas permit to express the magnitude of the azimuth angle $|\Delta\alpha|$ as a function of the frequency of vibrations f. This dependence is shown in Fig. 3. The resonance frequencies correspond to the roots

$$\lambda_1 = 0.430, \quad \lambda_2 = 1.470, \quad \lambda_3 = 2.416$$

and the frequencies in cycles per second equal

$$f_1 = 11.4, \quad f_2 = 20.6, \quad f_3 = 26.2$$

Translated by T.L.